Complexity Bounds of Iterative Linearization Algorithms for Discrete-Time Nonlinear Control

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Soon on ArXiv, paper and code available on request
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Machine Learning and Optimization (MLOpt) Seminar
02/11/2022

Nonlinear Control Problems

Continuous-Time Control problem

- System driven by dynamics $\dot{x}(t) = \bar{f}(x(t), u(t))$
- Minimize cost $\bar{h}(x(t), t)$ over $t \in [0, T]$ for $x(0)$ fixed

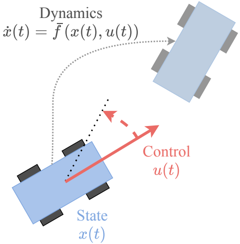
Discrete-Time Control Problem

- Discretize dynamics as $x_{t+1} = f(x_t, u_t)$
- Minimize costs $h_t(x_t)$ over $t \in \{0, \ldots, \tau\}$ for x_0 fixed

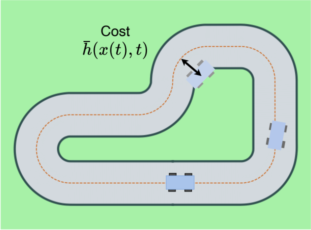
Algorithms Principle

Current controls $u_0, \ldots, u_{\tau-1}$ with trajectory $x_0, \ldots, x_\tau$

1. Linearize dynamics f around x_t, u_t
2. Take quadratic approx. of the costs h_t around x_t
3. Compute optimal policies for the resulting LQ pb
4. Update controls using the optimal policies



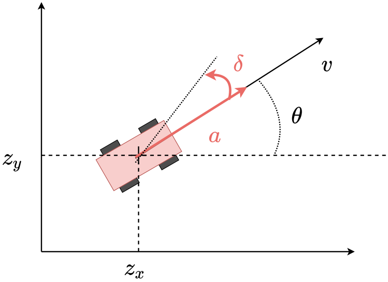
Dynamics of a car



Tracking objective

Autonomous Car Racing

Simple model of a car

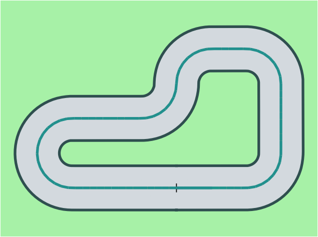


$$x = (z_x, z_y, \theta, v), \qquad u = (\delta, a)$$

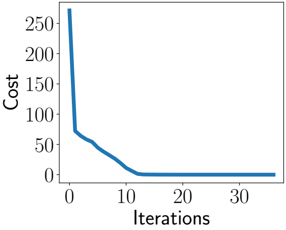
$$\dot{z}_x = v \cos\theta \qquad \dot{\theta} = v \tan(\delta)$$

$$\dot{z}_y = v \sin\theta \qquad \dot{v} = a$$

Algo. converges *fast* to *optimal trajectory*



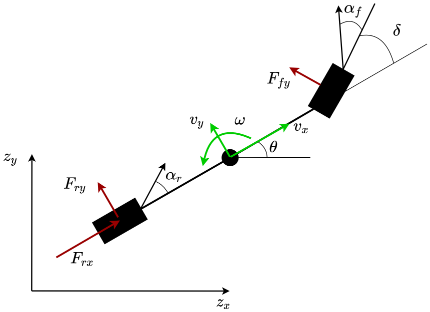
Optimized trajectory horizon $\tau = 100$



Convergence of the algorithm

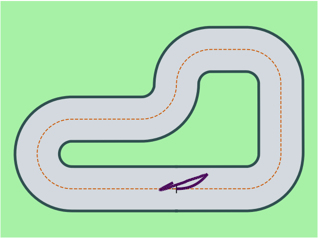
Autonomous Car Racing

Bicycle model of a car (Liniger et al. 2015)

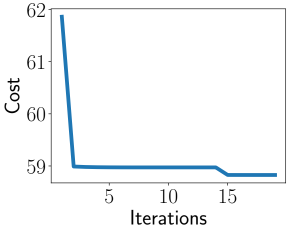


Models tire forces (highly non-linear)

Unclear whether the algorithm succeeded...



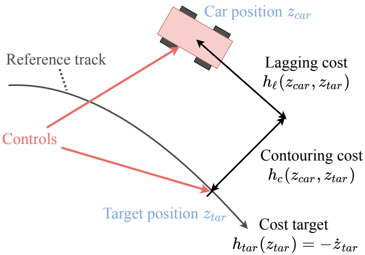
Optimized trajectory horizon $\tau = 100$

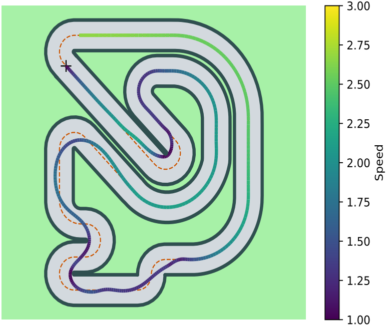


Convergence of the algorithm

Autonomous Car Racing

Contouring objective (Liniger et al. 2015)

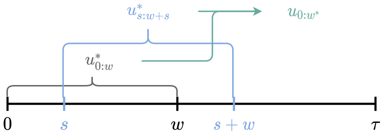




Optimized traj. with MPC & contouring

Model predictive control

Optimize trajectory incrementally on overlapping windows



What could be an optimal window size?
How many iterations per window?

Objectives and Outline

Questions

1. What are sufficient conditions to ensure global convergence?
2. What are the worst-case complexity bounds of these algorithms?
3. How does the discretization scheme impact global convergence guarantees?

Related work

- Sufficient optimality conditions in continuous-time (Mangasarian 1966)
  $\rightarrow$ Translatable in discrete-time, requires convexity of implicitly defined functions
- Local convergence of Differential Dynamic Programming
  (Polak 2011, Murray & Yakowitz 1984, Liao & Shoemaker 1991)
- Local convergence of generalized Gauss-Newton
  e.g. (Yamashita & Fukushima 2001, Diehl & Messerer 2019)
- Global convergence of regularized Gauss-Newton a.k.a. Levenberg-Marquardt
  e.g. (Bergou et al. 2020)

Plan

Discrete-Time Nonlinear Control Problems

Continuous-Time Control problem

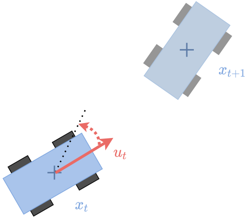
$$\min_{x(t),u(t)} \quad \int_0^T \bar{h}(x(t),t)$$
$$\text{s.t.} \quad \dot{x}(t) = \bar{f}(x(t),u(t)), \; x(0) = \bar{x}_0$$

Discrete-Time Control Problem

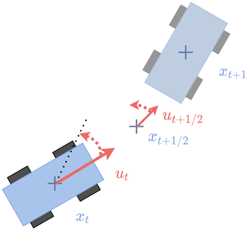
$$\min_{\substack{x_0;\ldots;x_\tau \\ u_0;\ldots;u_{\tau-1}}} \quad \sum_{t=1}^{\tau} h_t(x_t)$$
$$\text{s.t.} \quad x_{t+1} = f(x_t,u_t), \; x_0 = \bar{x}_0$$

Discretization schemes:

$$\text{Euler:} \quad f(x_t,u_t) = x_t + \Delta \bar{f}(x_t,u_t)$$
$$\text{Multi-step:} \quad f(x_t,u_t) = x_{t+1}$$
$$\text{s.t.} \quad x_{t+(s+1)/k} = x_s + \Delta \bar{f}(x_{t+s/k}, u_{t+s/k})$$
$$\dim(u_t) = k \dim(u(t))$$



Euler discretization



2-step discretization

Nonlinear Control Algorithms for Discrete-Time Control Problems

Forward Given a sequence of controls $u_0, \ldots, u_{\tau-1}$

a. Compute associated trajectory $x_{t+1} = f(x_t, u_t)$

b. Record linear expansions $\ell_f^{x_t, u_t}$ of the dynamics f around x_t, u_t

c. Record quadratic expansions $q_{h_t}^{x_t}$ of the costs around x_t

Backward Compute optimal policies π_t for the regularized linear-quadratic control problem

$$\min_{\substack{y_0, \ldots, y_\tau \\ v_0, \ldots, v_{\tau-1}}} \quad \sum_{t=1}^{\tau} q_{h_t}^{x_t}(y_t) + \frac{\nu}{2} \sum_{t=0}^{\tau-1} \|v_t\|_2^2$$

$$\text{s.t} \quad y_{t+1} = \ell_f^{x_t, u_t}(y_t, v_t), \quad y_0 = 0$$

by back-propagating the cost-to-go functions, starting from $c_\tau = q_{h_\tau}$,

$$c_t(y_t) = q_{h_t}(y_t) + \min_{v_t} \left\{ \frac{\nu}{2} \|v_t\|_2^2 + c_{t+1}(\ell_f^{x_t, u_t}(y_t, v_t)) \right\}$$

Roll-out Update the iterates as $u_t^{\text{next}} = u_t + v_t$

where v_t are computed by rolling-out the policies along either

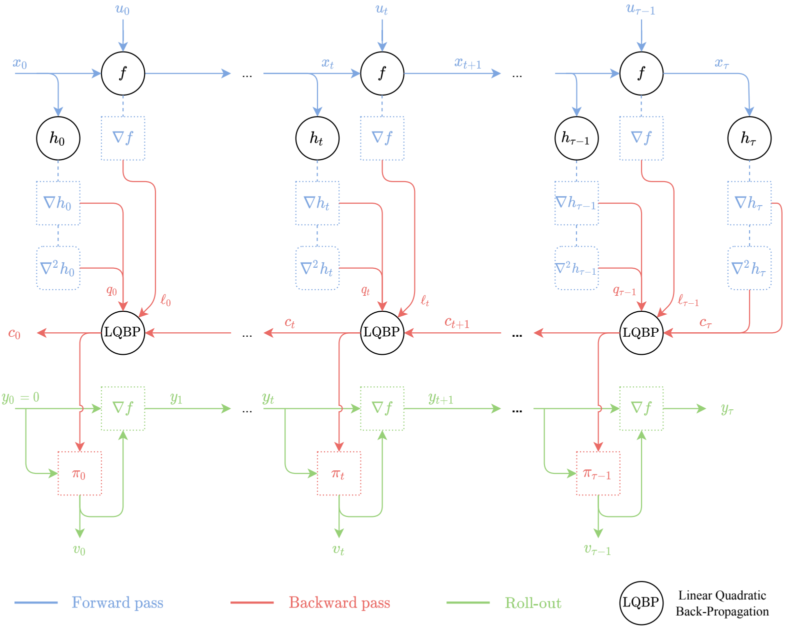
- the linearized dynamics → Iterative Linear Quadratic Regulator (ILQR) (Li & Todorov 2007)

$$v_t = \pi_t(y_t) \quad y_{t+1} = \ell_f^{x_t, u_t}(y_t, v_t)$$

- the original dynamics → Iterative Differential Dynamic Programming (IDDP) (Tassa et al. 2012)

$$v_t = \pi_t(y_t) \quad y_{t+1} = f(x_t + y_t, u_t + v_t) - f(x_t, u_t)$$

ILQR Computational Scheme



Optimization Viewpoint

Objective for $\boldsymbol{u} = (u_0; \ldots; u_{\tau-1})$

$$\mathcal{J}(\boldsymbol{u}) = \sum_{t=1}^{\tau} h_t(x_t)$$
$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t), \quad x_0 = \bar{x}_0$$

Algorithms

$$\boldsymbol{u}^{(k+1)} = \boldsymbol{u}^{(k)} + \mathrm{LQR}_{\nu_k}(\mathcal{J})(\boldsymbol{u}^{(k)}) \qquad \text{(ILQR)}$$
$$\boldsymbol{u}^{(k+1)} = \boldsymbol{u}^{(k)} + \mathrm{DDP}_{\nu_k}(\mathcal{J})(\boldsymbol{u}^{(k)}) \qquad \text{(IDDP)}$$

where $\mathrm{LQR}_{\nu_k}(\mathcal{J})(\boldsymbol{u}^{(k)})$, $\mathrm{DDP}_{\nu_k}(\mathcal{J})(\boldsymbol{u}^{(k)})$ are oracles returning a direction computed by dynamic programming as presented before for a regularization ν_k

Oracles Computational Complexities

$$O(\tau(\dim(x) + \dim(u))^3)$$

→ linear w.r.t. to horizon τ as the oracles exploit the dynamical structure

Simple car example with k steps scheme:
$\tau = 100$, $\dim(x) = 4$, $\dim(u) = 2k$ → leading dimension τ

Plan

Objective Decomposition

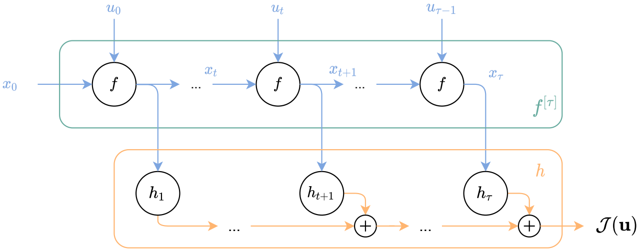
Control of τ steps of f for $\boldsymbol{u} = (u_0; \ldots; u_{\tau-1})$

$$f^{[\tau]}(x_0, \boldsymbol{u}) = (x_1; \ldots; x_\tau)$$
$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t)$$

Total cost for $\boldsymbol{x} = (x_1, \ldots, x_\tau)$

$$h(\boldsymbol{x}) = \sum_{t=1}^{\tau} h_t(x_t)$$

Composite objective $\quad \mathcal{J}(\boldsymbol{u}) = h(f^{[\tau]}(\bar{x}_0, \boldsymbol{u}))$



Sufficient Condition for Global Convergence

Idea

- Known sufficient condition for global conv. of 1st order methods, for $c > 0$,

$$\|\nabla \mathcal{J}(\boldsymbol{u})\|_2^2 \geq c(\mathcal{J}(\boldsymbol{u}) - \mathcal{J}^*) \qquad \text{(GDom)}$$

- Here consider that the total cost h is gradient dominated with constant $\mu > 0$
- Then $\mathcal{J}$ satisfies (GDom) if the control of τ steps of f satisfies

$$\forall \boldsymbol{u} \quad \sigma_{\min}(\nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})) \geq \sigma > 0 \qquad \text{(Suff Cond)}$$

  where $\sigma_{\min}(A) = \inf_{\|z\|>0} \|Az\|_2/\|z\|_2$ is the minimal singular value of A

- Indeed, denoting $\boldsymbol{x} = f^{[\tau]}(\bar{x}_0, \boldsymbol{u})$

$$\|\nabla \mathcal{J}(\boldsymbol{u})\|_2^2 = \|\nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u}) \nabla h(\boldsymbol{x})\|_2^2 \geq \sigma^2 \|\nabla h(\boldsymbol{x})\|_2^2 \geq \sigma^2 \mu (h(\boldsymbol{x}) - h^*) = \sigma^2 \mu (\mathcal{J}(\boldsymbol{u}) - \mathcal{J}^*)$$

Sufficient Condition for Global Convergence

Interpretation

$$\sigma_{\min}(\nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})) > 0 \iff \boldsymbol{\lambda} \mapsto \nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})\boldsymbol{\lambda} \text{ is injective}$$
$$\iff \boldsymbol{v} \mapsto \nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})^\top \boldsymbol{v} \text{ is surjective}$$

Here $\boldsymbol{y} = \nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})^\top \boldsymbol{v}$ is the linearization of the trajectories given as

$$y_{t+1} = \nabla_{x_t} f(x_t, u_t)^\top y_t + \nabla_{u_t} f(x_t, u_t)^\top v_t, \quad y_0 = 0$$

So $\sigma_{\min}(\nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})) > 0$ if the linearization of the trajectories are *surjective*

How to verify this condition from f only?

Sufficient Condition for Global Convergence

Lemma

If the linearization, $v \to \nabla_u f(x,u)^\top v$, of l_f-Lip. cont. dynamics f is surjective,

$$\forall x, u, \quad \sigma_{\min}(\nabla_u f(x,u)) \geq \sigma_f > 0, \tag{Surj}$$

then the linearization of the trajectories, $\boldsymbol{v} \to \nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})^\top \boldsymbol{v}$, is surjective,

$$\forall \boldsymbol{u} \quad \sigma_{\min}(\nabla_{\boldsymbol{u}} f^{[\tau]}(\bar{x}_0, \boldsymbol{u})) \geq \frac{\sigma_f}{1 + l_f} > 0,$$

Note: (Surj) requires $\dim(u_t) \geq \dim(x_t)$, yet generally $\dim(u(t)) < \dim(x(t))$
$\to$ possible by using multi-step discretization schemes s.t. $\dim(u_t) = k \dim(u(t))$

Sufficient Condition for Global Convergence

Conclusion

- If the costs h_t are gradient dominated and f has surjective linearizations, s.t.

$$\begin{aligned} \min_{u_t, x_t} \quad & h_{t+1}(x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} = f(x_t, u_t) \end{aligned}$$

  can be solved by e.g. a gradient descent

  $\rightarrow$ the objective $\mathcal{J}$ is gradient dominated

  $\rightarrow$ the problem can be solved by e.g. a gradient descent

- However, we are not considering gradient descent...

Plan

Nonlinear Control Problems and Algorithms

Objective Analysis

Convergence Analysis of Iterative Linear-Quadratic Approximations

Discretization Scheme and Convergence Guarantees

Assumptions

Problem

$$\min_{\boldsymbol{u}} \left\{ \mathcal{J}(\boldsymbol{u}) = h(g(\boldsymbol{u})) \right\}, \text{ where } g(\boldsymbol{u}) = f^{[\tau]}(\bar{x}_0, \boldsymbol{u}), \quad h(\boldsymbol{x}) = \sum_{t=1}^{\tau} h_t(x_t)$$

Assumptions

- costs h_t
  - μ_h-strongly convex $\rightarrow$ same for total cost h
  - L_h-smooth $\rightarrow$ same for total cost h
  - M_h-smooth Hessian $\rightarrow$ same for total cost h
- dynamic f
  - l_f-Lip. continuous, L_f smooth $\rightarrow$ g is l_g-Lip.continous, L_g-smooth

$$l_g \leq l_f S, \qquad L_g \leq L_f S (l_f S + 1)^2 \qquad \text{where } S = \sum_{t=0}^{\tau-1} l_f^t$$

  - $\sigma_{\min}(\nabla_u f(x, u)) \geq \sigma_f > 0 \qquad \rightarrow \sigma_{\min}(\nabla g(\boldsymbol{u})) \geq \sigma_g = \sigma_f / (1 + l_f) > 0$

Convergence Analysis Viewpoint

ILQR as a generalized Gauss-Newton (Sideris & Bobrow 2005)

- Overall ILQR minimizes a quadratic approx. of h on top of a linear approx. of g
- So it can be summarized as

$$
\begin{aligned}
\mathrm{LQR}_\nu(\mathcal{J})(\boldsymbol{u}) &= \arg\min_{\boldsymbol{v}} q_h^{g(\boldsymbol{u})}(\ell_g^{\boldsymbol{u}}(\boldsymbol{v})) + \frac{\nu}{2}\|\boldsymbol{v}\|_2^2 \\
&= -(\nabla g(\boldsymbol{u})\nabla^2 h(g(\boldsymbol{u}))\nabla g(\boldsymbol{u})^\top + \nu \mathrm{I})^{-1}\nabla g(\boldsymbol{u})\nabla h(g(\boldsymbol{u}))
\end{aligned}
$$

which is a *regularized generalized Gauss-Newton method*

Contributions

- Analysis of regularized generalized Gauss-Newton method given h strongly convex, g with surjective Jacobians $\nabla g(\boldsymbol{u})^\top$
- Similar approach as (Nesterov 2007) for modified Gauss-Newton a.k.a. prox-linear Global conv. of prox-linear under error bound cond. $\rightarrow$ (Drusvyatskiy & Lewis 2018)

Convergence Analysis

Global convergence idea

- Select regularization ν to ensure sufficient decrease, i.e., for $\boldsymbol{v} = \mathrm{LQR}_\nu(\boldsymbol{u})$, $\boldsymbol{x} = g(\boldsymbol{u})$, $G = \nabla g(\boldsymbol{u})$, $H = \nabla^2 h(\boldsymbol{x})$,

$$\begin{aligned}\mathcal{J}(\boldsymbol{u}+\boldsymbol{v}) &\leq \mathcal{J}(\boldsymbol{u}) + q_h^{\boldsymbol{x}} \circ \ell_g^{\boldsymbol{u}}(\boldsymbol{v}) + \frac{\nu}{2}\|\boldsymbol{v}\|_2^2 \qquad \text{(Suff Dec)} \\ &= \mathcal{J}(\boldsymbol{u}) - \frac{1}{2}\nabla h(\boldsymbol{x})^\top G^\top (GHG^\top + \nu \mathrm{I})^{-1} G \nabla h(\boldsymbol{x})\end{aligned}$$

- Given that $\sigma_{\min}(G) \geq \sigma_g$, we have $\lambda_{\min}(G^\top G) > \sigma_g^2$, so

$$\mathcal{J}(\boldsymbol{u}+\boldsymbol{v}) - \mathcal{J}(\boldsymbol{u}) \leq -\frac{\sigma_g^2}{l_g^2 L_h + \nu}\|\nabla h(\boldsymbol{x})\|_2^2 \leq -\frac{\sigma_g^2 \mu_h}{l_g^2 L_h + \nu}(\mathcal{J}(\boldsymbol{u}) - \mathcal{J}^*)$$

$\rightarrow$ linear convergence ensured for constant ν satisfying (Suff Dec)

- Condition (Suff Dec) is ensured for $\nu(\boldsymbol{u}) = c(\|\nabla h(\boldsymbol{x})\|_2)$ with c increasing

$\rightarrow$ decreasing regularizations can be taken to get better rates

Convergence Analysis

Local convergence idea

- By standard linear algebra, for $\boldsymbol{x} = g(\boldsymbol{u})$, $G = \nabla g(\boldsymbol{u})$, $H = \nabla^2 h(\boldsymbol{x})$,

$$\begin{aligned}
\mathrm{LQR}_\nu(\mathcal{J})(\boldsymbol{u}) &= -(GHG^\top + \nu\,\mathrm{I})^{-1} G \nabla h(\boldsymbol{x}) \\
&= -G(HG^\top G + \nu\,\mathrm{I})^{-1} \nabla h(\boldsymbol{x}) && \text{(Push-Forward Identity)} \\
&= -G(G^\top G)^{-1}(H + \nu(G^\top G)^{-1})^{-1} \nabla h(\boldsymbol{x}) && (G^\top G \text{ invertible})
\end{aligned}$$

- So denoting $\boldsymbol{x}^{\text{next}} = g(\boldsymbol{u} + \boldsymbol{v})$ for $\boldsymbol{v} = \mathrm{LQR}_\nu(\mathcal{J})(\boldsymbol{u})$,

$$\boldsymbol{x}^{\text{next}} \approx g(\boldsymbol{u}) + \nabla g(\boldsymbol{u})^\top \boldsymbol{v} = \boldsymbol{x} - (\nabla^2 h(\boldsymbol{x}) + \nu(\nabla g(\boldsymbol{u})^\top \nabla g(\boldsymbol{u}))^{-1})^{-1} \nabla h(\boldsymbol{x}).$$

$\rightarrow$ Approximate Newton method on the trajectories $\boldsymbol{x}$ for $\nu \ll 1$

$\rightarrow$ Quadratic local convergence can be ensured for decreasing regularizations ν

Complexity Bound for ILQR

Theorem

Consider strongly convex, smooth, Hessian-smooth costs h_t and Lip. cont., smooth dynamics f with surjective linearizations, the ILQR algorithm equipped with $\nu(\boldsymbol{u}) = \bar{\nu}\|\nabla h(g(\boldsymbol{u}))\|_2$ for $\bar{\nu}$ large enough converges to accuracy ε in at most

$$\underbrace{4\theta_g(\sqrt{\delta_0} - \sqrt{\delta})}_{\text{slow conv.}} + \underbrace{2\rho_h \ln\left(\frac{\delta_0}{\delta}\right) + 2\alpha \ln\left(\frac{\theta_g\sqrt{\delta_0} + \rho_g}{\theta_g\sqrt{\delta} + \rho_g}\right)}_{\text{linear conv.}} + \underbrace{O(\ln\ln(\varepsilon))}_{\text{quad. conv.}}$$

iterations, each having a comput. complexity $O(\tau(\dim(x) + \dim(u))^3)$, where

- $\delta_0 = \mathcal{J}(\boldsymbol{u}^{(0)}) - \mathcal{J}^*$ *is the initial gap*
- $\delta = 1/(32\rho_h(\theta_h(1 + \sqrt{\rho_h}{\rho_g}^3/3) + \sqrt{\rho_h}\theta_g(1 + \rho_g\rho_h))^2)$ *is the gap of quadratic conv.*
- $\rho_h = L_h/\mu_h$ *is the cond. nb of the costs*
- $\rho_g = l_g/\sigma_g$ *is the cond. nb of the linearized traj.*
- $\theta_h = M_h/\mu_h^{3/2}$ *is the param. of self-concordance of the costs*
- $\theta_g = L_g/(\sigma_g^2\sqrt{\mu_h})$ *acts a self-concordance param. for the linear-quadratic decomp.*
- $\alpha = 4{\rho_g}^2(2{\rho_g}^2\theta_h/(3\theta_g) + \rho_h)$ *is another cond. nb*

Extended analysis

Theorem

Given smooth convex costs h_t s.t., for some $\mu > 0$ and $r \in [1/2, 1)$,

$$\|\nabla h(x)\|_2 \geq \mu_h^r (h(x) - h^*)^r$$

and Lip. cont., smooth dynamics f with surjective linearizations,
the ILQR algorithm converges globally with a complexity

$$O(\varepsilon^{2r-1}/(2r-1) + \delta_0^{1-r}/(1-r)), \quad \textit{i.e.} \quad O(\ln(\varepsilon) + \sqrt{\delta_0}) \text{ if } r = 1/2$$

Theorem

Given convex, smooth, Hessian-smooth, self-concordant cost h
and Lip. cont., smooth dynamics f with surjective linearizations,
the ILQR algorithm converges locally with a quadratic rate

- Precise rates given in the paper in terms of the cond nb defined before

Plan

Analysis of the Sufficient Condition

Multistep scheme

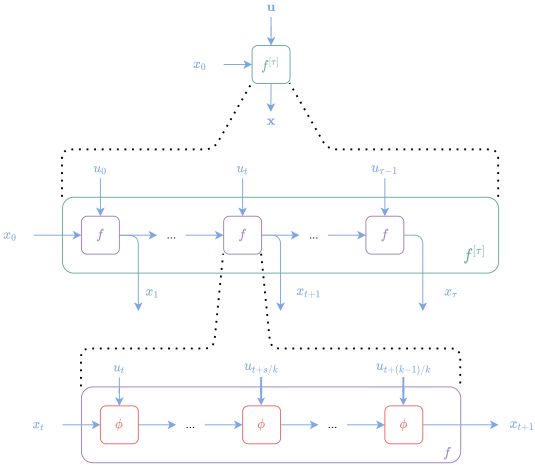
$$f(x_t, u_t) = x_{t+1}$$
$$x_{t+(s+1)/k} = x_s + \Delta \bar{f}(x_{t+s/k}, u_{t+s/k})$$

summarized as

$$f(x_t, u_t) = \phi^{\{k\}}(x_t, u_t)$$

Control in k steps of a dynamic ϕ
For $\boldsymbol{v} = (v_0, \ldots, v_{k-1})$,

$$\phi^{\{k\}}(y_0, \boldsymbol{v}) = y_k$$
$$\text{s.t.} \quad y_{s+1} = \phi(y_s, v_s)$$



Zooming in the dynamical structure

Analysis of the Sufficient Condition

Intepretation of the surjectivity of the linearizations

- Surjectivity of $\boldsymbol{w} \to \boldsymbol{z} = \nabla_u \phi^{\{k\}}(y_0, \boldsymbol{v})^\top \boldsymbol{w}$ is equivalent to the reachability of any state in k steps of the time-varying system

$$z_{s+1} = \nabla_{y_s} \phi(y_s, v_s)^\top z_s + \nabla_{v_s} \phi(y_s, v_s)^\top w_s, \quad z_0 = 0$$

- Reachability of original time-invariant dynamic ϕ has been studied
  $\to$ in continuous-time see e.g. (Isidori 1995, Sontag 2013)
  $\to$ in discrete-time (Grizzle 1986, Jakubczyk 1987, Jakubczyk & Sontag 1990)
- To tackle the reachability of the *linearizations*, we rather consider the existence of a feedback linearization scheme as detailed next

Static Feedback Linearization

Definition (Static Feedback Linearization for $\dim(y_t) = d, \dim(v_t) = 1$)

A dynamical system $y_{t+1} = \phi(y_t, v_t)$ is linearizable by static feedback
if there exists some diffeomorphism a and $b(y, \cdot)$ s.t.
the reparameterized sytem $z_t = a(y_t)$, $w_t = b(y_t, v_t)$ is linear of the form

$$z_{t+1}^{(i)} = z_t^{(i+1)} \text{ for all } i \in \{1, \ldots, d-1\}, \quad z_{t+1}^{(d)} = w_t,$$

Examples

- System driven by its acceleration, with $|\partial_{v_t}\psi(y_t, v_t)| > 0$

$$y_{t+1}^{(1)} = y_t^{(1)} + \Delta y_t^{(2)}, \quad y_{t+1}^{(2)} = y_t^{(2)} + \Delta\psi(y_t, v_t)$$

- System driven by its d^{th} derivative
- More generally, (Aranda-Bricaire et al. 1996) essentially showed that
  local feedback linearization $\iff$ reachability of any state by ϕ
  Proof is constructive and might be quantified

Multistep Schemes and Static Feedback Linearization

Idea

- If $z_{t+1}^{(i)} = z_t^{(i+1)}$ for all $i \in \{1, \dots, d-1\}$, $z_{t+1}^{(d)} = w_t$, then

$$\begin{array}{ccccc}
z^{(1)} & z^{(2)} & z^{(3)} & & w_0 \\
\vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & z^{(d)} & \dots & \vdots \\
\vdots & z^{(d)} & w_0 & & \vdots \\
z^{(d)} & w_0 & w_1 & & w_{d-1} \\
 & & & & \\
t=0 & t=1 & t=2 & \dots & t=d
\end{array}$$

$\rightarrow$ By considering d steps $z_d = w_{0:d-1}$,
$\rightarrow$ so the control in d steps of the reparameterized system is the identity
$\rightarrow$ so it clearly has surjective linearizations

- This property is kept under the diffeomorphisms a, b

Multistep Schemes and Static Feedback Linearization

Theorem (for $\dim(y_t) = d, \dim(v_t) = 1$)

If the system defined by $y_{t+1} = \phi(y_t, v_t)$ is linearizable by static feedback with transformations a and b that are Lipschitz-continuous and such that

$$\forall y \ \sigma_{\min}(\nabla a(y)) \geq \sigma_a > 0, \quad \inf_{y,v} \sigma_{\min}(\nabla_v b(y,v)) \geq \sigma_b > 0,$$

then the control in $k \geq d$ steps of the dynamic ϕ satisfies,

$$\inf_{y_0, \boldsymbol{v}} \sigma_{\min}(\nabla_{\boldsymbol{v}} \phi^{\{k\}}(y_0, \boldsymbol{v})) \geq \frac{\sigma_b}{l_a} \frac{1}{1 + (d-1) l_b / \sigma_a} > 0.$$

Take-away:

- Having access to the exact diffeomorphisms a, b may be intractable
- But showing their existence may be possible
  and global convergence guarantees follow

Conclusion

Outcomes

1. Identified a simple sufficient condition for global convergence
2. Linked this condition to known concepts in nonlinear control litterature such as reachability/feedback linearization
3. Provided detailed complexity bounds for ILQR and IDDP

Future directions

1. Incorporate costs and/or constraints on the controls
2. Explain better why IDDP works much better than ILQR
3. Quantify locality of feedback linearization scheme results
4. Derive cond. nb in terms of the discretization step Δ and the horizon τ

Thank you for your attention!

Complexity bound for IDDP

Idea
Analyze IDDP as an approximate ILQR similar as (Murray & Yakowitz 1984) for local conv.

Lemma

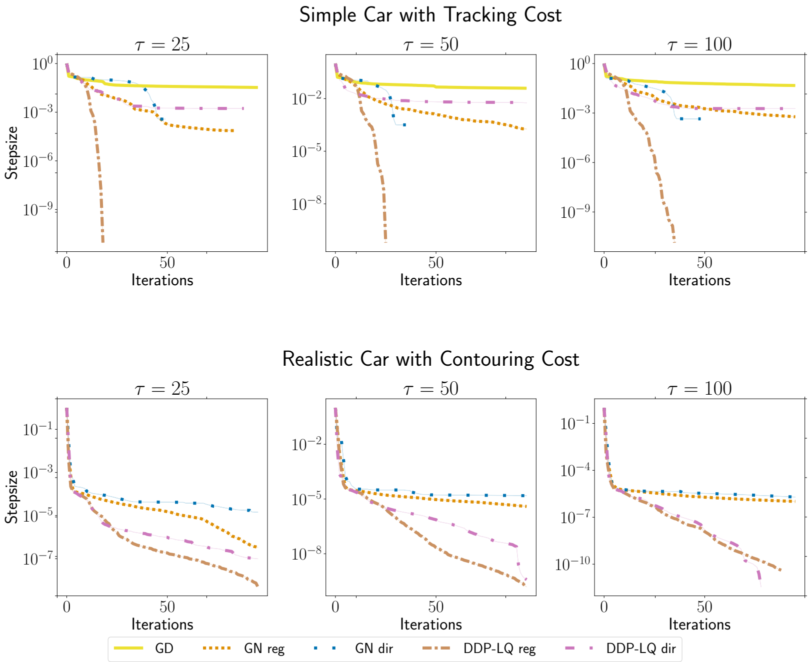
Given strongly convex, smooth, Hessian-smooth costs h_t,
Lip. cont., smooth dynamics f with surj. linearizations, there exists $\eta > 0$ s.t.

$$\forall \boldsymbol{u}, \nu \quad \| \mathrm{DDP}_\nu(\mathcal{J})(\boldsymbol{u}) - \mathrm{LQR}_\nu(\mathcal{J})(\boldsymbol{u})\|_2 \leq \eta \| \mathrm{LQR}_\nu(\mathcal{J})(\boldsymbol{u})\|_2^2$$

Theorem

Consider strongly convex, smooth, Hessian-smooth costs h_t
and Lip. cont., smooth dynamics f with surjective linearizations,
the IDDP algo. equipped with appropriate regularization
converges globally with a local quadratic rate.

Numerical Illustrations



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