Iterative Linearized Control from an Optimization Viewpoint

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INFORMS Annual Meeting 2019

Off-line Finite-Trajectory Control Problems

Dynamics

$$x_0 = \hat{x}_0, \quad x_{t+1} = \phi_t(x_t, u_t) \quad \text{for } t \in \{0, \ldots, \tau-1\}$$

Off-line Finite-Trajectory Control Problems

Deterministic Control Objective

$$\min_{\substack{u_0,\dots u_{\tau-1} \\ x_0,\dots,x_\tau}} \quad \sum_{t=0}^{\tau} h_t(x_t) + \sum_{t=0}^{\tau-1} g_t(u_t)$$

$$\text{subject to} \quad x_0 = \hat{x}_0, \quad x_{t+1} = \phi_t(x_t, u_t) \quad \text{for } t \in \{0, \dots, \tau-1\}$$

with $x_t \in \mathbb{R}^d$ states, $u_t \in \mathbb{R}^p$ controls

Off-line Finite-Trajectory Control Problems

Expected Control Objective

$$\min_{\substack{u_0,\ldots u_{\tau-1}\\ x_0,\ldots,x_\tau}} \quad \mathbb{E}_{w_0,\ldots w_{\tau-1}}\left[\sum_{t=0}^{\tau} h_t(x_t)\right] + \sum_{t=0}^{\tau-1} g_t(u_t)$$

subject to $\quad x_0 = \hat{x}_0, \quad x_{t+1} = \phi_t(x_t, u_t + w_t) \quad$ for $t \in \{0, \ldots, \tau-1\}$

with $x_t \in \mathbb{R}^d$ states, $u_t \in \mathbb{R}^p$ controls, $w_t \sim \mathcal{N}(0, \sigma^2 \, \mathrm{I}_p)$ noises.

Off-line Finite-Trajectory Control Problems

Risk-Sensitive Control Objective

$$\min_{\substack{u_0,\dots u_{\tau-1}\\ x_0,\dots,x_\tau}} \quad \frac{1}{\theta}\log \mathbb{E}_{w_0,\dots w_{\tau-1}}\left[\sum_{t=0}^{\tau} \exp(\theta h_t(x_t))\right] + \sum_{t=0}^{\tau-1} g_t(u_t)$$

subject to $\quad x_0 = \hat{x}_0, \quad x_{t+1} = \phi_t(x_t, u_t + w_t) \quad$ for $t \in \{0, \dots, \tau-1\}$

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Off-line Finite-Trajectory Control Problems

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with $x_t \in \mathbb{R}^d$ states, $u_t \in \mathbb{R}^p$ controls, $w_t \sim \mathcal{N}(0, \sigma^2 \operatorname{I}_p)$ noises.

Interpretation for $\theta \to 0$

$$\frac{1}{\theta} \log \mathbb{E}_w \exp[h(x,w)] = \mathbb{E}_w[h(x,w)] + \frac{\theta}{2} \mathbb{V}\mathrm{ar}_w[h(x,w)] + \mathcal{O}(\theta^2)$$

Iterative Linearization Algorithms

Idea: For *linear* dynamics and *quadratic* costs, previous control problems can be solved exactly by dynamic programming

Iterative Linearization Algorithms

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Iterative Linear Quadratic Regulator (ILQR) (Li & Todorov 2004)

Nonlinear control $\longrightarrow$ **Iterative linearization**

$$\min_{\substack{u_0,\ldots,u_{\tau-1}\\ x_0,\ldots x_\tau}} \sum_{t=0}^{\tau} h_t(x_t) + g_t(u_t)$$

$$\text{s.t.} \quad x_{t+1} = \phi_t(x_t, u_t), \quad x_0 = \hat{x}_0$$

1. Around current $x_t^{(k)}, u_t^{(k)}$, solve

$$\min_{\substack{v_0,\ldots,v_{\tau-1}\\ y_0,\ldots y_\tau}} \sum_{t=0}^{\tau} y_t^\top H_{t,x} y_t + v_t^\top G_{t,u} v_t$$

$$\text{s.t.} \quad y_{t+1} = \Phi_{t,x} y_t + \Phi_{t,u} v_t, \quad y_0 = 0$$

2. Line-search $u_t^{(k+1)} = u_t^{(k)} + \alpha v_t^*$

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Questions

Control - Do iterative linearizations (ILQR, ILQG, ILEQG) converge?

- Can we improve the line-search in a principled way?

Optim - How algorithms of practitioners translate as optim. algo.?

Optimization Viewpoint

Noiseless Trajectory is a function of $\bar{u} = (u_0, \ldots, u_\tau - 1)$, as

$$\tilde{x}(\bar{u}) = (\tilde{x}_1(\bar{u}); \ldots; \tilde{x}_\tau(\bar{u}))$$

$$\tilde{x}_1(\bar{u}) = \phi_0(\hat{x}_0, u_0), \qquad \tilde{x}_{t+1}(\bar{u}) = \phi_t(\tilde{x}_t(\bar{u}), u_t).$$

Optimization Viewpoint

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Control problems read

$$\min_{\bar{u}} \quad h(\tilde{x}(\bar{u})) + g(\bar{u}) \qquad \text{(Deterministic)}$$
$$\min_{\bar{u}} \quad \mathbb{E}_{\bar{w}}[h(\tilde{x}(\bar{u} + \bar{w}))] + g(\bar{u}) \qquad \text{(Expected)}$$
$$\min_{\bar{u}} \quad \frac{1}{\theta} \log \mathbb{E}_{\bar{w}}[\exp \theta h(\tilde{x}(\bar{u} + \bar{w}))] + g(\bar{u}) \qquad \text{(Risk-Sensitive)}$$

with $h(\bar{x}) = \sum_{t=1}^{\tau} h_t(x_t)$, $g(\bar{u}) = \sum_{t=0}^{\tau-1} g_t(u_t)$

Note: For noisy dynamics, we do not have access to the objective

Plan

Iterative Linearized Algorithms for Deterministic Dynamics

Iterative Linearized Algorithms for Noisy Dynamics

Differential Dynamic Programming Interpretation

Numerical Illustrations

Iterative Linearizations as Model-Minimization

Deterministic case

Approx. Trajectory for a given deviation $\bar{v}$ from current $\bar{u} = \bar{u}^{(k)}$,

$$\tilde{x}(\bar{u} + \bar{v}) \approx \tilde{x}(\bar{u}) + \nabla \tilde{x}(\bar{u})^\top \bar{v}$$

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Approx. Objective $f_\theta(\bar{u} + \bar{v}) \approx m_{f_\theta}(\bar{u} + \bar{v}; \bar{u})$ with

$$m_{f_\theta}(\bar{u}+\bar{v}; \bar{u}) \triangleq q_h\big(\tilde{x}(\bar{u})+\nabla \tilde{x}(\bar{u})^\top \bar{v}; \tilde{x}(\bar{u})\big) + q_g(\bar{u} + \bar{v}; \bar{u}),$$

where $q_h(\bar{x} + \bar{y}; \bar{x}) \triangleq h(\bar{x}) + \nabla h(\bar{x})^\top \bar{y} + \bar{y}^\top \nabla^2 h(\bar{x}) \bar{y}/2.$

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ILQR

1. Find $\bar{v}^* = \arg\min_{\bar{v}} m_{f_\theta}(\bar{u}+\bar{v}; \bar{u})$ *(LQR by dyn. prog.)*
2. Find by line-search α s.t. $\bar{u}^{(k+1)} = \bar{u}^{(k)} + \alpha \bar{v}^*$

Iterative Linearizations from Optimization Viewpoint

Deterministic Case

Regularized Iterative Linearizations with step-size γ_k

$$\bar{u}^{(k+1)} = \arg\min_{\bar{v}} m_{f_\theta}(\bar{u}+\bar{v};\bar{u}) + \frac{1}{2\gamma_k}\|\bar{v}\|_2^2 \quad \textit{(Lin. Quad. dyn. prog.)}$$

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Consequences (Roulet, Srinivasa, Drusvyatskiy & Harchaoui 2019)

For h, g quadratics, ϕ_t bounded, Lipschitz, smooth

1. Provide criterion to choose step-size γ_k
2. Show convergence to stationary point of the regularized algo.
3. Propose accelerated variant under suitable assumptions
4. Propose faster implementation for final state cost

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Optimization Remarks

1. Gradient, Gauss-Newton, Newton step have approx. same cost
2. ILQR is Gauss-Newton, we propose Regularized Gauss-Newton and an accelerated variant

Plan

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Noisy case

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$$\tilde{x}(\bar{u} + \bar{v} + \bar{w}) \approx \tilde{x}(\bar{u}) + \nabla \tilde{x}(\bar{u})^\top (\bar{v} + \bar{w})$$

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$$\begin{aligned} m_{f_\theta}(\bar{u}+\bar{v}; \bar{u}) \triangleq \mathbb{E}_{\bar{w}}\, q_h\big(\tilde{x}(\bar{u})+\nabla\tilde{x}(\bar{u})^\top \bar{v}+\nabla\tilde{x}(\bar{u})^\top \bar{w}; \tilde{x}(\bar{u})\big) \\ + q_g(\bar{u} + \bar{v}; \bar{u}), \end{aligned} \tag{Expected}$$

where $q_h(\bar{x} + \bar{y}; \bar{x}) \triangleq h(\bar{x}) + \nabla h(\bar{x})^\top \bar{y} + \bar{y}^\top \nabla^2 h(\bar{x}) \bar{y}/2.$

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ILQG or ILEQG

1. Find $\bar{v}^* = \arg\min_{\bar{v}} m_{f_\theta}(\bar{u}+\bar{v}; \bar{u})$ *(LQG or LEQG by dyn. prog.)*
2. Find by line-search α s.t. $\bar{u}^{(k+1)} = \bar{u}^{(k)} + \alpha \bar{v}^*$

Iterative Linearizations from Optimization Viewpoint

Regularized Iterative Linearizations with step-size γ_k

$$\bar{u}^{(k+1)} = \arg\min_{\bar{v}} m_{f_\theta}(\bar{u}+\bar{v}; \bar{u}) + \frac{1}{2\gamma_k}\|\bar{v}\|_2^2 \quad \textit{(Lin. Quad. dyn. prog.)}$$

$\rightarrow$ how to perform line-search on γ_k ?

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Surrogate Cost

$$\hat{f}_\theta(\bar{u}) = \underbrace{\mathbb{E}_{\bar{w}}\, h(\tilde{x}(\bar{u}) + \nabla\tilde{x}(\bar{u})^\top \bar{w})]}_{\hat{\eta}_\theta(\bar{u})} + g(\bar{u}) \qquad \text{(Expected)}$$

Iterative Linearizations from Optimization Viewpoint

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$\rightarrow$ how to perform line-search on γ_k ?

Surrogate Cost

$$\hat{f}_\theta(\bar{u}) = \underbrace{\frac{1}{\theta}\log \mathbb{E}_{\bar{w}} \exp[\theta h(\tilde{x}(\bar{u}) + \nabla \tilde{x}(\bar{u})^\top \bar{w})]}_{\hat{\eta}_\theta(\bar{u})} + g(\bar{u})$$

(Risk-Sensitive)

Iterative Linearizations from Optimization Viewpoint

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Surrogate Cost

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(Risk-Sensitive)

Consequences (Roulet, Fazel, Srinivasa & Harchaoui 2019)

For h, g quadratics, ϕ_t bounded, Lipschitz, smooth

1. Show that regularized iterative linearizations minimize $\hat{f}_\theta(\bar{u})$
2. $\hat{\eta}_\theta(\bar{u})$ can be computed analytically $\rightarrow$ access to line-search
3. Prove convergence to a near-stationary point of $\hat{f}_\theta$ for small γ_k

Plan

ILQR by Differential Dynamic Programming

Idea (Tassa et al. 2012)

1. Approximate Bellman equation at each step as a quadratic
2. Use computed gains to move along the true trajectory

Difference with Iterative Linearizations

1. Use same forward pass to approx. dynamics and costs
2. Use same backward pass to compute gains
3. Use roll-out on the true trajectory (not the linearized one)

Optim. Interpretation (Roulet, Srinivasa, Drusvyatskiy & Harchaoui 2019)

1. Recursive model-minimization
2. Variants possible for e.g. control of multiple trajectories

Plan

Iterative Linearized Algorithms for Deterministic Dynamics

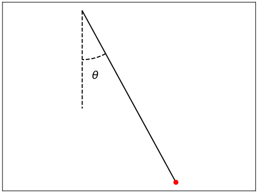
Iterative Linearized Algorithms for Noisy Dynamics

Differential Dynamic Programming Interpretation

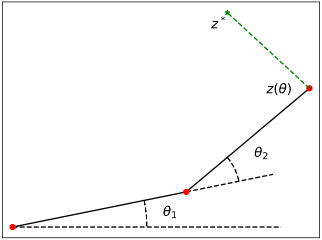
Numerical Illustrations

Numerical Illustrations for Risk-Sensitive Cost

Settings



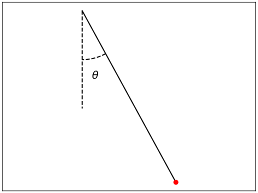
(a) Pendulum param. by θ.



(b) Two-link arm param. by θ_1, θ_2.

Numerical Illustrations for Risk-Sensitive Cost

Settings



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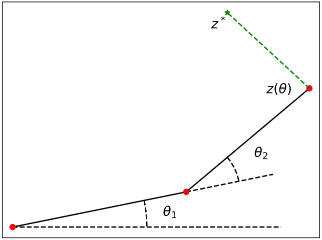

(b) Two-link arm param. by θ_1, θ_2.

Discretized Dynamics from $\ddot{z}(t) = f(z(t), \dot{z}(t), u(t))$

$$\begin{aligned} z_{t+1} &= z_t + \delta \dot{z}_t \\ \dot{z}_{t+1} &= \dot{z}_t + \delta f(z_t, \dot{z}_t, u_t), \end{aligned} \rightarrow (z_t, \dot{z}_t) = \phi_t((z_t, \dot{z}_t), u_t)$$

Numerical Illustrations for Risk-Sensitive Cost

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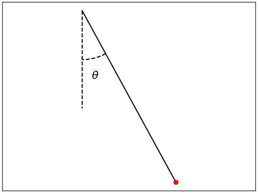

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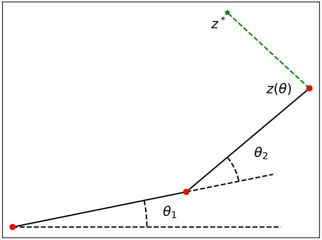

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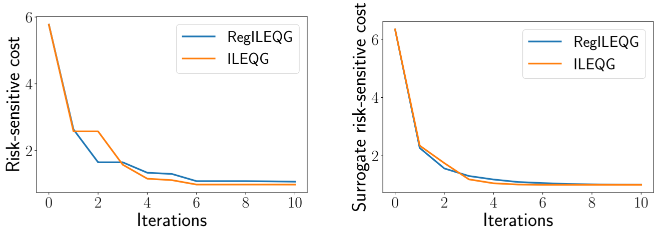
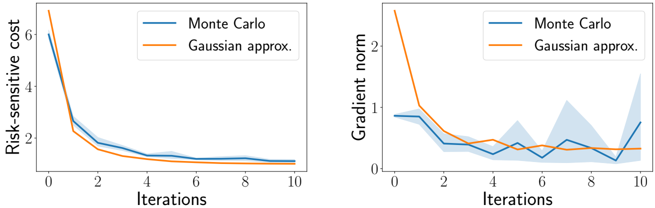
Robustness Test

$$\begin{aligned} z_{t+1} &= z_t + \delta \dot{z}_t \\ \dot{z}_{t+1} &= \dot{z}_t + \delta f(z_t, \dot{z}_t, u_t + \rho \mathbf{1}(t = t_w)), \end{aligned}$$

Numerical Illustrations for Risk-Sensitive Cost

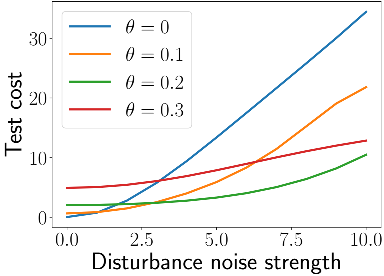


Convergence of ILEQG, RegILEQG



Approximations by surrogate risk-sensitive cost vs Monte-Carlo

Numerical Illustrations for Risk-Sensitive Cost



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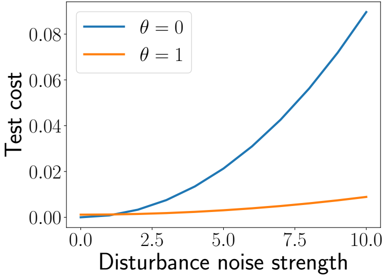

(b) Two-link arm.

Robustness of controllers against disturbance noise.

Code available at https://github.com/vroulet/ilqc

Conclusion

Outcomes

1. Translated control algorithms into optimization algorithms
2. Proposed regularized iterative linearizations
   with convergence guarantees to stationary points
3. Clarified line-searches using surrogate costs

Thank you ! Questions ?

References

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Tassa, Y., Erez, T. & Todorov, E. (2012), Synthesis and stabilization of complex behaviors through online trajectory optimization, *in* '2012 IEEE/RSJ International Conference on Intelligent Robots and Systems', IEEE, pp. 4906–4913.